Test of population mean when variance is not known.

$H_0: \mu = \mu_0$ when σ^2 unknown under the normal setup.
$x_1, x_2, \ldots, x_n \sim N(\mu, \sigma^2)$

You see here two parameters μ, σ^2 but both of the parameters are unknown. So we have to use the unbiased, sufficient statistics for both μ and σ^2.
Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from $N(\mu, \sigma^2)$.

$\bar{x}$ = sample mean.

$$= \frac{1}{n} \sum x_i$$

$$\text{sample variance} = s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

we known $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$

and $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

Also, $\bar{x}$ and s^2 are independently distributed.

To test $H_0: \mu = \mu_0$ and $H_1: \mu > \mu_0$ when σ^2 unknown we use the test statistic based on the function of $\bar{x}, s^2$ both.

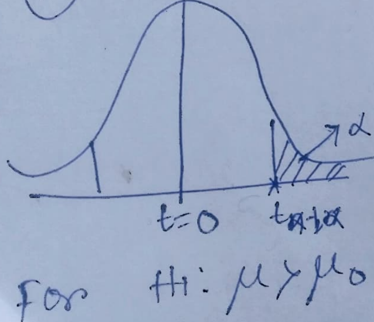
$$t = \frac{\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}}{\sqrt{\frac{(n-1)s^2}{\sigma^2(n-1)}}} = \frac{N(0,1)}{\sqrt{\chi^2/n-1}} \quad \text{——— (A)}$$

Also, the numerator and denominator are distributed independently.

$\Rightarrow t = \frac{\sqrt{n}(\bar{x}-\mu)}{s} \sim t_{n-1}$

(A) $\sim t_{n-1}$.

$\boxed{\text{Remember}}$ t distribution is symmetric ranging from 0 to ∞. (only difference from normal distribution is



under $t = t_{calculated} = \frac{\sqrt{n}(\bar{x}-\mu_0)}{s}$

For $H_1: \mu > \mu_0$, we reject H_0 if $t_{calculated} > \underbrace{t_{n-1,\alpha}}_{\text{cut off point}}$

$\boxed{\text{Remember.}}$ ① when n is given, for $\alpha = 5\%$ and 1% you've to see the t table.

② The naming of the cutoff point will be the same way we did in $H_0: \mu = \mu_0$, σ^2 known test.

i.e. $Pr_{H_0}(t > c) = \alpha$

$\Rightarrow$ $c = t_{\alpha, n-1}$

powers $= P_{H_1}\left[\frac{\sqrt{n}(\bar{x}-\mu_0)}{s} > t_{n-1,\alpha} \middle| \mu > \mu_0\right]$

$= P_{H_1}\left[\frac{\sqrt{n}(\bar{x}-\mu+\mu-\mu_0)}{s} > t_{n-1,\alpha} \middle| \mu > \mu_0\right]$

$= P_{H_1}\left[\frac{\sqrt{n}(\bar{x}-\mu)}{s} > t_{n-1,\alpha} - \frac{\sqrt{n}(\mu-\mu_0)}{s}\right]$

$= P\left[t > t_{n-1,\alpha} - \frac{\sqrt{n}(\mu-\mu_0)}{s}\right]$

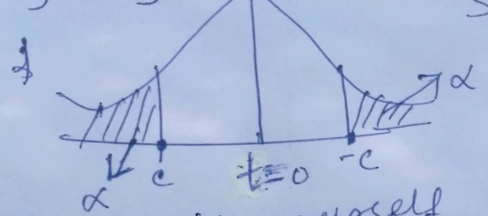
$= 1 - F\left(t_{n-1,\alpha} - \frac{\sqrt{n}(\mu-\mu_0)}{s}\right)$

II) For $H_0: \mu = \mu_0$ against $H_1: \mu < \mu_0$ (left tail test)

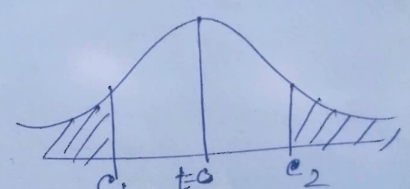
Everything remains same except the cutoff point.

So we reject H_0 if.

$\frac{\sqrt{n}(\bar{x}-\mu)}{s} = t_{calculated} < -t_{\alpha, n-1}$



$\boxed{\text{Power}}$: do it yourself.



III) For $H_1: \mu \neq \mu_0$

we reject H_0 if $\frac{\sqrt{n}(\bar{x}-\mu_0)}{s} < c_1$

Or $\frac{\sqrt{n}(\bar{x}-\mu_0)}{s} > c_2$ where cutoff point

if $t_{cal} < c_1$ or $t_{cal} > c_2$

c_1 and c_2 will be determined from size condition.

$Pr[t_{cal} < c_1] + P[t_{cal} > c_2] = \alpha$

considering the equal tailed test, we get.

$c_1 = t_{1-\alpha/2, n-1}, \quad c_2 = t_{\alpha/2, n-1}$

As t is symmetric $t_{1-\alpha/2} = -t_{\alpha/2, n-1}$.

power $= Pr_{H_1}\left[\frac{\sqrt{n}(\bar{x}-\mu_0)}{s} > t_{\alpha/2, n-1}\right] + Pr_{H_1}\left[\frac{\sqrt{n}(\bar{x}-\mu_0)}{s} < -t_{\alpha/2, n-1}\right]$

$= Pr.\left[t > t_{\alpha/2, n-1} - \frac{\sqrt{n}(\mu-\mu_0)}{s}\right] + Pr\left[t < -t_{\alpha/2, n-1} - \frac{\sqrt{n}(\mu-\mu_0)}{s}\right]$

$= Pr\left[t > t_{\alpha/2, n-1} - \frac{\sqrt{n}(\mu-\mu_0)}{s}\right] + Pr\left[t > t_{\alpha/2, n-1} + \frac{\sqrt{n}(\mu-\mu_0)}{s}\right]$

$\boxed{\text{Remember}}$ This test statistic is called Student's t statistic.